Exercises Number Theory I

1 Divisibility

Beginner

1.1 Show that 900 divides 10!.

1.2 The product of two numbers, neither of which is divisible by 10, is 1000. Find the sum of the numbers.

1.3 Find all positive integers n such that n divides $n^2 + 3n + 27$.

Intermediate

1.4 Show that:

(a) $5 \cdot 17 \mid 5^2 \cdot 17 + 3 \cdot 5 \cdot 9 + 5 \cdot 3 \cdot 8$

(b) $n(n+m) \mid 3mn^2 + amn^2 + 3n^3 + an^3$

1.5 Find three three-digit positive integers, using nine different digits, so that the product ends in four zeros.

1.6 (a) Find all positive integers that have exactly 41 divisors, and are divisible by 41.

(b) Find all positive integers that have exactly 42 divisors, and are divisible by 42.

Olympiad

1.7 Find all positive integers n such that $n + 1 \mid n^2 + 1$.

1.8 Prove: for each positive integer n there exist n consecutive integers such that none of them is prime.

1.9 Prove that there are infinitely many positive integers n such that $2n$ is a square, $3n$ is a third power and $5n$ is a fifth power.

2 gcd and lcm

Beginner

2.1 (IMO 59) Show that the following fraction is in lowest terms.

$$\frac{21n+4}{14n+3}$$

2.2 Find all pairs of positive integers (a, b) with

$$\mathrm{lcm}(a, b) = 10 \gcd(a, b)$$

Intermediate

2.3 Every positive integers greater than 6 is the of two coprime positive integers greater than 1.

2.4 We call positive integers a and b *friends* if $a \cdot b$ is a square. Show that if a and b are friends, then a and $lcm(a, b)$ are friends.

Olympiad

2.5 Let m and n be two positive integers which sum to a prime number. Show that m and n are coprime. Seien m und n zwei natürliche Zahlen, deren Summe eine Primzahl ist. Zeige, dass m und n teilerfremd sind.

2.6 (Canada 97) Find the number of pairs of positive integers (x, y) with $x \leq y$ which satisfy the following equalites:

$$\gcd(x, y) = 5! \text{ and } \mathrm{lcm}(x, y) = 50!$$

3 Upper bounds

Beginner

3.1 We call a rectangle *nice* if its sides are integer lengths and the area and circumference are integers as well. Determine all nice rectangles.

3.2 Find all pairs (x, y) of positive integers with

$$\frac{1}{x} + \frac{2}{y} = 1.$$

Intermediate

3.3 We call a cuboid *nice* if its sides are integer lengths and the volume and surface are integers as well. Determine all nice cuboids.

3.4 Find all triple (x, y, z) of positive integers with

$$\frac{1}{x} + \frac{2}{y} - \frac{3}{z} = 1.$$

3.5 Find all positive integers n such that $n^2 + 1$ is a divisor of $n^7 + 13$.

Olympiad

3.6 Show that the equation

$$y^2 = x(x+1)(x+2)(x+3)$$

has no solution in the positive integers.

3.7 Find all positive integers x with

$$x! = x^2 + 11x - 36$$

3.8 (IMO 98) Determine all pairs (a, b) of positive integers such that $ab^2 + b + 7$ divides $a^2b + a + b$.

4 Exercises from Previous Olympiads

Questions from past olympiads are excellent preparation materials: they are of course at the correct level of difficulty, and additionally, all solutions can be found on `www.imosuisse.ch`. But make sure to give the problems a good try before looking up the solutions!

4.1 **(Preliminary round 2012, 1.)** Determine all pairs (m, n) of positive integers such that mn divides $(m+1)(n+2)$.

4.2 **(Preliminary round 2004, 1.)** Find all positive integers a,b and n such that the following equation holds:

$$a! + b! = 2^n$$

4.3 **(Preliminary round 2005, 3.)** Let m and n be coprime positive integers. Show that $m^3 + mn + n^3$ and $mn(m+n)$ are coprime.

4.4 **(Preliminary round 2011, 2.)** Find all positive integers n such that n^3 is the product of three positive divisors of n.

4.5 **(Preliminary round 2006, 1.)** Determine all triples (p, q, r) of prime numbers such that

$$|p-q|, |q-r|, |r-p|$$

are all primes.

4.6 **(Preliminary round 2008, 4.)** Determine all positive integers n such that the number of positive divisors of n are equal to the third smallest positive divisor of n.

4.7 **(Preliminary round 2013, 4.)** Determine all pairs (m, n) of positive integers such that

$$(m+1)! + (n+1)! = m^2 n^2$$

.