Induction Exercises

1 Exercises

Beginner

1.1 Show that for all natural numbers n, $n^2 + n$ is even.

1.2 Suppose $n \geq 3$ Show that in an n-gon, the sum of the interior angles is $(n-2) \cdot 180$ degrees.

1.3 Show that for all $n \in \mathbb{N}$, the following hold:

(a) $$1 + 3 + 5 + \cdots + (2n-1) = n^2$$

(b) $$1 + 2 + 4 + 8 + \cdots + 2^n = 2^{n+1} - 1$$

(c) $$1^2 + 2^2 \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

(d) $$\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n-1}{n!} = \frac{n! - 1}{n!}$$

(e) $$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$$

1.4 Find all natural numbers n such that $3^n > n!$.

1.5 Show that for all $n \in \mathbb{N}$

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$$

1.6 Show that for all $n \in \mathbb{N}$, $7^{2n} - 2^n$ is divisible by 47.

Intermediate

1.7 Show that any natural number can be expressed as the sum of some distinct powers of 2.

1.8 Show that the numbers $1007, 10017, 100117, 1001117, \ldots$ are all divisible by 53.

1.9 n lines divide the plane into some number of regions, where a region is a part of the plane with boundaries determined by the lines (with some regions having infinite area), and no line going through its interior. Show that there are at most $\frac{n^2+n+2}{2}$ regions.

1.10 **Preliminary Round 2016, 2** Quirin has n blocks with heights 1 to n and would like to place them in a row so that his cat can walk over them from left to right. His cat will begin on the block furthest to the left, and can go from one block to the next if and only if the next block is either shorter or one unit taller than the one its standing on. In how many ways can Quirin place the blocks given these restrictions?
Example: For $n = 5$*,* $3-4-5-1-2$ *is an allowable placement, but* $1-3-4-5-2$ *is not.*

1.11 n points in the plane are coloured either blue or red. We connect all red points with all blue points. Show that we can't have more than $\frac{n^2+n-2}{4}$ connections.

1.12 How many subsets of $\{1, 2, \ldots, n\}$ are there that don't contain any two consecutive elements?

1.13 Suppose $n \geq 2$. In a restaurant, n people are sitting in a row. There are three meals to choose from, and no one will eat the same meal as the person immediately to their left or right. In how many ways can the people order their food?

1.14 We have n books of different sizes, and three trays on which we can place the books. In the beginning, all books are lying on tray 1 sorted by size with the smallest on top. In one move, we are allowed to move the top book on one plate to another plate. However, we can never place a larger book on a smaller book. How many moves are required to move all books from plate 1 to plate 3?

Olympiad

1.15 Suppose $n \geq 6$. Show that one can divide a square into n square pieces.

1.16 Suppose $n \in \mathbb{N}$. Consider a $2^n \times 2^n$ chessboard, from which a tile has been removed. Show that it's possible to tile this board with L-triominoes.

1.17 A car with an empty fuel tank is on a round racetrack, and there are n fuel stops placed around the track. Divided up amongst the fuel stops is just enough fuel to get around the track. Should the car reach a fuel stop, it will get all of the fuel there. Show that there is a fuel stop from which the car can begin such that it can get all the way around the track without running out of fuel.

1.18 There are $n \geq 2$ candies in a bowl. Alice and Bob play a game: In one turn, one is allowed to take a positive number of candies out of the bowl, but no more than half of the candies in the bowl. Alice and Bob alternate turns, and Alice begins. One loses if after one's turn there is one candy left in the bowl. For what n does Bob have a winning strategy?