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Preliminary round 2020

Solutions

Preliminary remark: A complete solution is worth 7 points. For every problem, up to 2 points can be deducted from a correct solution for (minor) flaws. Partial marks are attributed according to the marking schemes.

Below you will find the elementary solutions known to correctors. Alternative solutions are presented in a complementary section at the end of each problem. Students are encouraged to use any methods at their disposal when training at home, but should be wary of attempting to find alternative solutions using methods they do not feel comfortable with under exam conditions as they risk losing valuable time.

G1) Let k be a circle with center O. Let A, B, C and D be four distinct points on k in this order such that AB is a diameter of k. The circumcircle of the triangle COD intersects AC again in P. Show that OP and BD are parallel.

Solution 1 (Louis), Angle chasing: We will show that $\angle ODB = \angle POD$, which proves that the lines are parallel.

As O is the centre of the circle, we have that $OD = OB$ and so the triangle ODB is isoceles in O. So $\angle ODB = \angle OBD$. As O lies on the segment AB, $\angle OBD = \angle ABD$. The points A, B, C, D are concyclic so we also have $\angle ABD = \angle ACD = \angle PCD$. Finally, the points C, O, D, P are all on a circle and so $\angle PCD = \angle POD$. Therefore, we have

$$\angle ODB = \angle POD.$$

and we are done.

Marking Scheme

For partial solutions, the following points are available ($\leq$ 4P):

(a) Forward points:

Remark: For the following angle equalities, angles marked on a drawing **are not** sufficient. The relation should be explicitly stated elsewhere.

- +1P: establish an angle equality in the cyclic quadrilaterals $ABCD$ **or** $CPDO$ (eg. $\angle PCD = \angle POD$ or $\angle DBA = \angle ACD$)
- +1P: establish an angle equality combining the two quadrilaterals $ABCD$ **and** $CPDO$ (eg. $\angle PCD = \angle DBA$)
- +1P: establish an angle qualitiy using that O is the centre of the circle $ABCD$ (eg. $\angle ODB = \angle OBD$)

(b) Backward points:

- +1P: reformulate the conclusion in terms of angles (eg. $\angle POD = \angle ODB$ or $\angle DBA = \angle POA$)

Alternative solutions

Solution 2 (Tanish), Nine-point circle:

Let X be $AC \cap BD$ and Q be the second intersection of BD and (COD). We will prove that (COD) is the nine-point circle of AXB. But this is clear - O is the midpoint of AB, D is the base of the altitude dropped from B (Thales) and C is the base of the altitude dropped from A (Thales). It follows that P and Q are the midpoints of AX and BX respectively, and $XPOQ$ is a parallelogram.

Solution 3 (Tanish), Inversion:

Let us invert about k. AC is sent to the circumcircle of OAC; the circumcircle of OAD is sent to DC. This means that P' is the intersection of these two. OP is sent to the line OP' and BC is sent to BC, so it suffices to prove that $OP' \parallel BD$. This is equivalent to $\angle AOP' = \angle ABD$. Since $\angle ABD = 2\angle AOD$, we simply have to prove OP' bisects $\angle AOD$, or that (as $AO = OD$), OP' bisects $\angle AP'D$. However this last statement is clearly true when you consider the circumcircle of $AODP'$ as $\angle OP'A$ subtends the arc $\overset{\frown}{OA}$ and $\angle OP'D$ subtends the arc $\overset{\frown}{OC}$, which are of equal length as $OA = OC$.

Note: OP is the same line as OP' and therefore you could equally prove that OP bisects AOD or APD, but this is not as trivial.

G2) Let ABC be a triangle with $AB > AC$. The angle bisectors at B and C meet at point I inside the triangle ABC. The circumcircle of the triangle BIC intersects AB again in X and AC again in Y. Show that CX is parallel to BY.

Solution 1 (Arnaud), Angle chasing :

We will show that $\angle ACX = \angle AYB$. This condition is sufficient to conclude that the lines are parallel. We will suppose for our proof that X is between A and B and that Y is not betweeen A and C.

As CI is the bisector of $\angle ACB$, we have $\angle ACI = \angle ICB$. The points $ICYB$ are concyclic, so $\angle ICB = \angle IYB$. We now have that

$$\angle ACI = \angle IYB.$$

As $ICYB$ is a cyclic quadrilateral, we also have that $\angle CYI = \angle CBI$. The line IB is the bisector of $\angle CAB$, so $\angle CBI = \angle IBA$. The point X is between A et B, so $\angle IBA = \angle IBX$. Finally, as the points $XBCI$ are all on a circle, we have $\angle IBX = \angle ICX$. So

$$\angle CYI = \angle ICX.$$

In conclusion

$$\angle ACX = \angle ACI + \angle ICX = \angle IYB + \angle CYI = \angle CYB = \angle AYB.$$

For the last equality, we used the fact that C is between A and Y.

Solution 2 (Arnaud), Thales' intercept theorem :

In this solution we will show that $AX/AB = AC/AY$, which, by Thales' intercept theorem implies that XC and BY are parallel.

As $XICB$ is cyclic and X is between A and B, we have $\angle AXI = \angle BCI$. As CI is the bisector of the angle in C, we have $\angle BCI = \angle ICA$. Additionally,

$$\angle AXI = \angle ACI.$$

Similarly, using that $XICB$ is cyclic and IB is the bisector of the angle in B, we have

$$\angle ICX = \angle IBX = \angle IBC = \angle IXC.$$

Juxtaposing these two statements and using that X is between A and B, we obtain:

$$\angle AXC = \angle AXI + \angle IXC = \angle ACI + \angle ICX = \angle ACX,$$

and so the triangle ACX is isoceles in A; in other words, $AX = AC$. There are multiple ways to conclude. For example:

(a) As $XCYB$ is cyclic, power of a point gives $AX \cdot AB = AC \cdot AY$. Since $AX = AC$, we have $AB = AY$. Thus we have $AX/AB = AC/AY$.

(b) As $XCYB$ is cyclic, we have $\angle XBY = \angle ACX$ and $\angle AXC = \angle CYB$. As the triangle AXC is isoceles in A, $\angle AXC = \angle ACX$ and so $\angle XBY = \angle CYB$. Furthermore, $\angle ABY = \angle AYB$. So the triangle ABY is isoceles in A and so $AB = AY$. We conclude as before.

Marking Scheme:

Partial solutions are marked as follows: Additive partial points ($\leq$ 3P):

(a) Forward points:

- +1P: establish an angle relation in a cyclic quadrilateral with 4 points from $\{X, I, C, Y, B\}$
- +1P: establish an angle relation in **another cyclic quadrilateral** with 4 points from $\{X, I, C, Y, B\}$
- +1P: An angle equality which uses both a cyclic quadrilateral with 4 points from $\{X, I, C, Y, B\}$ and one of the bisectors $\{CI, BI\}$
- +2P: Show that the center of the circle passing through B, X, I, C, Y corresponds to the intersection of AI with the circumcircle of ABC (Incenter-Excenter lemma)

(b) Backward points:

- +0P: Reformulate the conclusion in terms of angles (eg. $\angle ACX = \angle AYB$) or length-ratios ($AX/AB = AC/AY$)
- +1P: Reformulate the conclusion in terms of angles *with* decomposition (eg. $\angle ACI = \angle IYB$ and $\angle ICX = \angle AYI$) or an equality of lengths ($AX = AC$ and $AB = AY$)

Partial results (non-additive with the preceding points, $\geq$ 4P):

(a) Partial results worth 4P:

- Show that ACX is isoceles and therefore $AX = AC$, or
- show that ABY is isoceles and therefore $AB = AY$

(b) Partial results worth 5P:

- Show that $AX = AC$ and $AB = AY$
- Show that $\angle ACX = \angle AYB$ or any other angle relation that allows one to immediately conclude that the lines are parallel.

Remark: It is not necessary to prove X is between A and B, or that C is between A and Y.

Alternative solutions

Solution 3 (Tanish), Incenter-Excenter Lemma:

The Incenter-Excenter Lemma tells us that if we prolong AI until it intersects the circumcircle of ABC at I_A, I_A happens to be the circumcenter of BIC. There are multiple ways to conclude, one of which is to take the reflection across the line AI_A, which sends A to A, B to Y and C to X (as the lines AB and AC are swapped and the circle (BIC) is preserved) and it immediately follows that $BY \parallel CX$.

K1) Consider a white 5×5 square composed of 25 unit squares. How many different ways are there to colour one or more unit squares black such that the resulting black area is a rectangle?

Solution 1 (Tanish), Counting by opposite vertices: Let us consider the possible vertices of any rectangle we create in the grid, which is the group of 36 points in a 6×6 square. Choose one of these as the first vertex of our rectangle. If we now choose the opposite corner, then the pair of these two points sufficiently define the rectangle. The opposite corner cannot be in the same column or line, so is one of 25 other points. Counting over all 36 points, we have 36×25 pairs of points. But we have counted each rectangle 4 times - there are two different pairs of opposite vertices which we each counted twice (they are ordered pairs) - so we divide our product by 4 to obtain $9 \times 25 = 225$.
Remark (Jana): It's also possible to count the possible choices of opposite vertices in the following way: First we choose two distinct points in the 6×6-grid. There are $\binom{36}{2} = 18 \cdot 35$ possibilities to do that. Next we have to subtract those where both points lie in the same row or column. Since there are 6 rows and 6 columns and for each one we have $\binom{6}{2}$ possible choices, the number we subtract is $12 \cdot \binom{6}{2} = 12 \cdot 15$. Noticing that this way, each proper rectangle is counted twice, we obtain our final result $\frac{1}{2} \cdot (18 \cdot 35 - 12 \cdot 15) = 225$.

Solution 2 (Tanish), Counting by height and width: Let us consider the number of possible rectangles of dimension $a \times b$, where a is the height and b is the width. The top-left square can be any of the squares in a $(6-a) \times (6-b)$ rectangle in the top left, as otherwise the rectangle will not be fully contained in the square. Therefore the number of possibilities is

$$\sum_{a=1}^{5}(6-a)\sum_{b=1}^{5}(6-b) = \sum_{a=1}^{5}(6-a) \cdot 15 = 15 \cdot 15 = 225$$

Symmetrically, you could also have considered the possible locations of the top-right, bottom-left or bottom-right square in terms of the dimensions of the rectangle.

Solution 3 (Tanish), Counting by rows/columns of sides: Our rectangle is well defined by the choice of two rows and two columns of the grid (the rows representing the top and bottom side and the columns representing the left and right side). We have 6 rows and 6 columns to choose from, so the total number of possibilities is $\binom{6}{2} \cdot \binom{6}{2} = 15^2 = 225$.

Solution 4 (Tanish), Counting by top-right square: Let us number our squares with coordinates (x, y), with $(1, 1)$ being the bottom-left square and $(5, 5)$ the top-right square. Now let us look at all the rectangles whose top right corner is (a, b). The rectangle is well-defined by the choice of a top-right and bottom-left square, so all we have to do is choose another square $(< a, < b)$, of which there are $a \cdot b$ possibilities. Therefore, summing over all possible (a, b) we have

$$\sum_{a=1}^{5} a \sum_{b=1}^{5} b = \sum_{a=1}^{5} a \cdot 15 = 15 \cdot 15 = 225$$

Symmetrically, this proof also works when considering the top-left, bottom-right or bottom-left square.

Solution 5 (Tanish), Counting by smallest side: Let us count all the rectangles whose smaller side is of length x. Inclusion-Exclusion tells us that we should count the number of rectangles of height x and width $\geq x$ plus the number of width x and height $\geq x$ minus the number of width *and* height x. By the result in proof 2 we already know the number of rectangles of dimension $a \times b$ is $(6-a)\cdot(6-b)$ so we have

$$\begin{aligned}
\sum_{x=1}^{5}\left((2\cdot\sum_{y=x}^{5}\Big((6-y)\cdot(6-x)\Big)-(6-x)(6-x)\right) &= \sum_{x=1}^{5}(6-x)(1+2+\ldots(6-x)\ldots... +2+1)\\
&= \sum_{x=1}^{5}(6-x)(6-x)^2\\
&= \sum_{x=1}^{5}(6-x)^3\\
&= 125+64+27+8+1=225
\end{aligned}$$

A similar proof is possible when you count by largest side.

All 5 of the proofs above can be generalised to an $n \times n$ square.

Marking scheme

- 1P: State the right answer 225 (this can be added to the other marks)

The following are non-additive.

- 1P: Any sensible attempt to enumerate the squares (e.g coordinates) or seperating the rectangles into disjoint sets.
- 3P: Finding a way of uniquely describing a rectangle (e.g by choosing opposite vertices, rows and columns, position of top right square and side lengths etc.)
- 4P: Justified expression that counts each rectangle the same number of times.
- 4P: Separating the rectangles into disjoint sets and counting the number of rectangles in each set.
- 5P: Justified formula for the exact number of rectangles

Without a correct formula, at most 4 points can be awarded. To obtain full mark, non-trivial sums must be explicitely computed.

Alternative solutions

Solution 6 (Tanish), General proof by induction using bijections: Let us prove the general result that the number of rectangles in a $n \times n$ square is $\sum_{i=1}^{n} i^3$. For the base case, there is clearly only 1 possible rectangle in a 1×1 square. Now suppose the proposition holds true for the case n. Now expand the grid to an $(n+1) \times (n+1)$ grid by adding a column on the right and a row on top. For every rectangle possible in the case n we now modify it by moving its top-right corner one column up and to the right. This application represents a bijection between the rectangles in the case n and the rectangles in the case $n+1$ without a side of length 1 (this can be verified by seeing that both the application and its inverse are surjective). It remains to count the "new" rectangles, or those with at least one side of length 1. These are well-defined by the choice of a square and then a square in the same row or column to represent the "ends" of the rectangle (the second square can be the same as the first one!), and this method counts every rectangle twice. We have n^2 choices for the first square and $2n$ choices for the second square after that (n in the same row, n in the same column) giving a total of $2n^3$ new rectangles, which we divide by 2, to give n^3.
Note: It is possible to use this same method for solution 5 to count the number of rectangles of smaller side x, as these rectangles are well defined by the choice of two $x \times x$ squares in the same columns or rows (again, counting each rectangle twice).

K2) The village of Roche has 2020 residents. One day, the famous mathematician Georges de Rham makes the following observations:

- Every villager knows someone else with the same age as them.
- For any group of 192 people in the village, there are always at least three of them that have the same age.

Prove that there must exist a group of 22 villagers that all have the same age.

Solution: We prove there are at most 95 different possible ages. Suppose there are 96 or more different ages amongst them. Since each villager knows someone else with the same age as them, we can find two villagers for each of 96 different ages, forming a group of 192 villagers where no group of three with the same age exists, contradicting de Rham's second observation. Knowing that there are 95 different ages, we can now apply the pigeonhole principle to get that there is an age group represented at least $\lceil \frac{2020}{95} \rceil = 22$ times. Note that 96 different ages also yields this value, so the limiting case is groups of 194 people.

Marking Scheme: The first two are non-additive.

- 2P: Show that it suffices to prove that there are at most k different ages for some $k \in \{95, 96\}$
- 1P: Show that it suffices to prove that there are at most k different ages for some $k \in \{92, 93, 94\}$
- 5P: Prove that there are at most k ages for some $k \in \{95, 96\}$

Z1) If $p \geq 5$ is a prime number, let q denote the smallest prime number such that $q > p$ and let n be the number of positive divisors of $p+q$ (1 and $p+q$ included).

a) Prove that no matter the choice of p, the number n is always at least 4.

b) Find the actual minimal value m that n can reach among all possible choices for p. That is:

b_1) Give an example of a prime number p for which the value m is reached.

b_2) Prove that there is no prime number p for which n is smaller than m.

Solution: As $p \geq 5$ the two prime numbers p, q are both odd and so $p+q$ is even, implying $p+q$ is divisible by $1, 2, \frac{p+q}{2}, p+q$. Additionally, $p+q > 4$ so the four given divisors are distinct, proving part a).

For part b_1) testing small values of p shows that n doesn't seem to be inferior to 6 and that $p+q$ has 6 divisors when $p=5$, $q=7$ for example, where the divisors are $1, 2, 3, 4, 6, 12$.

For b_2) the key idea is that $\frac{p+q}{2}$ is not prime. Since p and q are two consecutive primes and $p < \frac{p+q}{2} < q$, this number cannot be prime. There are a few subcases we have to evaluate:

- If $\frac{p+q}{2}$ is divisible by two distinct primes $s, t \neq 2$, then $p+q$ has at least 8 divisors: $1, 2, s, t, 2s, 2t, st, 2st$.
- If $\frac{p+q}{2}$ is divisible by $2, s$ with $s \neq 2$ a prime number, then $p+q$ has at least 6 divisors: $1, 2, 4, s, 2s, 4s$.
- If $\frac{p+q}{2}$ is divisible by s^2 with $s \neq 2$ a prime number, then $p+q$ has at least 6 divisors: $1, 2, s, 2s, s^2, 2s^2$.
- If $\frac{p+q}{2}$ is a power of 2, then so is $p+q$. As $32 = 2^5$ has 6 distinct divisors, and every subsequent power of 2 is divisible by these 6 divisors, it suffices to prove $p+q$ cannot be a power of 2 inferior to 32. As $5 \leq p < q$, it follows that $p+q > 8$, and so the only remaining case is $p+q = 16$. For this we remark that $5+7 = 12 < 16$ and $7+11 = 18 > 16$, and taking $p > 7$ will yield an even higher sum and so $p+q = 16$ is impossible.

Marking Scheme

- 1P : Proving $n \geq 4$.
- 1P : Stating the correct value m with an explicit example.
- 2P : Proving that $\frac{p+q}{2}$ is not prime.
- +3P : Finishing the proof.

If in the case distinction at the end of the proof, one possible situation is not covered, this will not be considered a minor mistake and at most 5 points can be awarded.

Z2) Let p be a prime number and a, b, c and n positive integers with $a, b, c < p$ such that the three assertions

$$p^2 \mid a + (n-1) \cdot b, \qquad p^2 \mid b + (n-1) \cdot c, \qquad p^2 \mid c + (n-1) \cdot a.$$

hold. Show that n is not a prime number.

Solution 1 (Louis): The key idea here is to sum the three divisibility statements. We obtain:

$$p^2 \mid \big(a + (n-1) \cdot b\big) + \big(b + (n-1) \cdot c\big) + \big(c + (n-1) \cdot a\big) = n \cdot (a + b + c).$$

As $a, b, c < p$ we have that $a + b + c \leq 3p - 3 < 3p$ and so if $p \geq 3$ we obtain $a + b + c < p^2$. In particular, p^2 does not divide $a + b + c$ and so forcibly $p \mid n$. Consequently, we write $n = k \cdot p$ (and so n is a prime number if and only if $k = 1$). Using the first statment, we have that $p^2 \mid a + (kp - 1) \cdot b = a - b + kpb$. Notably, we must have that $p \mid a - b$, and as a, b are both strictly between 0 and p the only possibility is $a = b$. Using the first statement again we see that $p^2 \mid a + nb - b = nb$ and since b is coprime with p (as $b < p$), it follows that $p^2 \mid n$, implying n is not a prime number.

At this stage it is important to note that we stated at the start $p \geq 3$ and so we should also treat the case $p = 2$. Here, we have no choice but to take $a = b = c = 1$, and so substituting in the first divisibility relation yields directly $p^2 \mid n$, implying that n is not a prime number.

Solution 2 (David): As in the previous solution we first prove that $p \mid n$. From this it follows that n can be a prime number only if $n = p$. We therefore assume by contradiction that $n = p$. If we look at the first divisibility relation we necessarily have $p^2 \leq a + (n-1) \cdot b$. On the other hand, as $a, b < p$ it follows that $a + (n-1) \cdot b = a + (p-1) \cdot b < p + (p-1) \cdot p = p^2$. Therefore, $p^2 < p^2$, yielding the desired contradiction.

As in the previous solution we still need to treat the case $p = 2$. The same proof as before concludes the exercise.

Marking Scheme:

- 2P: Proving that $p^2 \mid n \cdot (a + b + c)$.
- +2P : Proving that $p \mid n$.
- +2P : Proving that $a = b$ **or** that $a + (n-1)b < p^2$ (or any other observation after which the proof is easily finished).
- +1P : Finish the proof.

1 point will be deducted if the solution is correct notwithstanding a finite amount of prime numbers p not being covered.